

# ON GROMOV'S CONJECTURE FOR TOTALLY NON-SPIN MANIFOLDS.

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**ABSTRACT.** Gromov's Conjecture states that for a  $n$ -manifold  $M$  with positive scalar curvature the macroscopic dimension of its universal covering  $\widetilde{M}$  satisfies the inequality  $\dim_{mc} \widetilde{M} \leq n-2$  [G2]. The conjecture was proved for some classes of spin manifolds [BD, Dr1]. Here we consider the Gromov Conjecture for totally non-spin manifolds. In particular we prove the conjecture for  $n$ -manifolds  $M$  with the virtually abelian fundamental groups.

Also we prove a weaker inequality  $\dim_{mc} \widetilde{M} \leq n-1$  for the universal coverings of positive scalar curvature  $n$ -manifolds whose fundamental groups are virtual duality groups satisfying the Rosenberg-Stolz conditions.

## 1. INTRODUCTION

The notion of macroscopic dimension was introduced by M. Gromov [G2] to study topology of manifolds that admit a positive scalar curvature (PSC) metric. We recall that the scalar curvature of a Riemannian  $n$ -manifold  $M$  is a function  $Sc_M : M \rightarrow \mathbb{R}$  which assigns to each point  $x \in M$  the sum  $Sc_x$  of the sectional curvatures over all 2-planes  $e_i \wedge e_j$  in the tangent space  $T_x M$  at  $x$  for some orthonormal basis  $e_1, \dots, e_n$ .

**1.1. Definition.** A metric space  $X$  has the macroscopic dimension  $\dim_{mc} X \leq k$  if there is a uniformly cobounded map  $f : X \rightarrow K$  to a  $k$ -dimensional simplicial complex. Then  $\dim_{mc} X = m$  where  $m$  is minimal among  $k$  with  $\dim_{mc} X \leq k$ .

We recall that a map of a metric space  $f : X \rightarrow Y$  is uniformly cobounded if there is a uniform upper bound on the diameter of preimages  $f^{-1}(y)$ ,  $y \in Y$ .

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**Gromov's Conjecture.** *The macroscopic dimension of the universal covering  $\widetilde{M}$  of a closed PSC  $n$ -manifold  $M$  satisfies the inequality  $\dim_{mc} \widetilde{M} \leq n - 2$  for the metric on  $\widetilde{M}$  lifted from  $M$ .*

The main examples supporting Gromov's Conjecture are  $n$ -manifolds of the form  $M = N \times S^2$ . They admit metrics with PSC in view of the formula  $Sc_{x_1, x_2} = Sc_{x_1} + Sc_{x_2}$  for the Cartesian product  $(X_1 \times X_2, \mathcal{G}_1 \oplus \mathcal{G}_2)$  of two Riemannian manifolds  $(X_1, \mathcal{G}_1)$  and  $(X_2, \mathcal{G}_2)$  and the fact that  $Sc_N$  is bounded while  $Sc_{S^2}$  can be arbitrary large. Note that the projection  $p : \widetilde{M} = \widetilde{N} \times S^2 \rightarrow \widetilde{N}$  is a uniformly cobounded map to a  $(n - 2)$ -dimensional manifold. Hence,  $\dim_{mc} \widetilde{M} \leq n - 2$ .

Since  $\dim_{mc} X = 0$  for every bounded metric space, the Gromov Conjecture holds trivially for simply connected manifolds. Thus, this conjecture is about manifolds with nontrivial fundamental groups. To what extend Gromov's Conjecture is a conjecture about groups? This is the question that we are trying to answer. We say that Gromov's Conjecture holds for a group  $\pi$  if it holds for manifolds with the fundamental group  $\pi$ . Thus, it makes sense to investigate Gromov's Conjecture for classes of groups. Clearly, the conjecture holds true for all finite groups. This paper establishes the Gromov Conjecture for the class of virtually abelian groups.

Dealing with PSC manifolds one has to consider three different cases: the case of spin manifolds, almost spin manifolds, and totally non-spin manifolds. We adopt the names *almost spin* for manifolds with the spin universal covering and *totally non-spin* for manifolds whose universal covering is non-spin.

We note that in the case of spin manifolds (as well as almost spin) there is index theory which provides a technique for attacking Gromov's Conjecture. There is no such technique available in the totally non-spin case since neither the manifold nor its universal covering has a K-theory fundamental class. This makes the totally non-spin case a notoriously difficult.

In this paper we manage to prove the Gromov's conjecture for virtual products of free groups in the totally non-spin case. For the almost spin manifolds it was established in our previous work. In fact for almost spin manifolds the Gromov Conjecture was proved for large classes of groups [Dr1] such as the virtually nilpotent groups, the arithmetic groups, the mapping class groups. At the moment we don't know how to cover these classes in the totally non-spin case.

The results of this paper are heavily based on our previous work. The first progress on Gromov's Conjecture in the spin case was made in [B1]. In [BD] we proved Gromov's Conjecture for *spin*  $n$ -manifolds whose fundamental group  $\pi$  virtually satisfies the Rosenberg-Stolz conditions (RS-conditions):

- The homomorphism  $ko_n(B\pi) \rightarrow KO_n(B\pi)$  induced by the transformation of spectra  $ko \rightarrow KO$  is a monomorphism;
- The Strong Novikov Conjecture holds for  $\pi$ : The analytic assembly map  $\alpha : KO_*(B\pi) \rightarrow KO_*(C^*(\pi))$  is a monomorphism, where  $C^*(\pi)$  is reduced  $C^*$  algebra of  $\pi$ .

In [Dr1] Gromov's Conjecture was proved for almost spin manifolds whose fundamental group is a virtual geometrically finite duality group that satisfies the coarse Baum-Connes Conjecture. We note here that the coarse Baum-Connes Conjecture implies the Strong Novikov Conjecture [Ro].

Gromov's Conjecture has a natural weak version:

**The Weak Gromov Conjecture.** *The macroscopic dimension of the universal covering  $\widetilde{M}$  of a closed PSC  $n$ -manifold  $M$  satisfies the inequality  $\dim_{mc} \widetilde{M} \leq n - 1$  for the metric on  $\widetilde{M}$  lifted from  $M$ .*

The Weak Gromov Conjecture first appeared in [G1] in the language of filling radii. Even the Weak Gromov Conjecture is out of reach, since it implies the Gromov-Lawson conjecture asserting that *a closed aspherical manifold cannot carry a PSC metric*. The latter is known to be a Novikov type conjecture [R2].

Here we prove the Weak Gromov Conjecture for totally non-spin manifolds whose fundamental groups are virtual duality groups that satisfy the RS-conditions. We do it by a reduction to the spin case. There are several important steps in that reduction. One of the steps which makes it all possible is the Jung-Stolz Theorem [RS]. Other important steps include the recent Gaifullin's realization theorem combined with Davis' theorem about Tomei manifolds and the use of the loop (string) homology.

In this paper we consider manifolds of dimension  $\geq 5$ . For 3-manifolds the Gromov Conjecture was proved in [GL]. The case of 4-manifolds should be treated differently.

## 2. PRELIMINARIES

Let  $\pi = \pi_1(K)$  be the fundamental group of a complex  $K$ . By  $u^K : K \rightarrow B\pi$  we denote a map that classifies the universal covering

$\widetilde{K}$  of  $K$ . We refer to  $u^K$  as a *classifying map* for  $K$ . We note that a map  $f : K \rightarrow B\pi$  is a classifying map if and only if it induces an isomorphism of the fundamental groups.

The following Rosenberg's theorem is the main tool for dealing with Gromov's Conjecture in the spin case.

**2.1. Theorem** (Rosenberg [R1, R2]). *Let  $[M]_{KO}$  denote the fundamental class of a closed spin  $n$ -manifold  $M$  in the  $KO$ -theory. Let  $\pi$  denote the fundamental group  $\pi_1(M)$ , then  $\alpha u_*^M([M]_{KO}) = 0$ , where the homomorphism  $u_*^M : KO_n(M) \rightarrow KO_n(B\pi)$  is induced by a classifying map  $u^M : M \rightarrow B\pi$ .*

Rosenberg's theorem implies that in the presence of the RS-conditions the element  $u_*^M([M]_{ko}) \in ko_n(B\pi)$  is an obstruction to the existence of a PSC metric on  $M$ .

The following result is the main tool in our reduction of the Gromov Conjecture from the totally non-spin case to the spin case.

**2.2. Theorem** (Jung-Stolz [RS]). *Suppose that  $N$  is a totally non-spin manifold with  $u_*^N([N]) = u_*([M])$  for some not necessarily connected manifold  $M$  with a positive scalar curvature and a map  $u : M \rightarrow B\pi$ . Then  $N$  admits a metric of positive scalar curvature.*

The following is well-known.

**2.3. Theorem** ([AB]). *Let  $p : L \rightarrow N$  be the canonical spin  $S^1$  bundle over a spin<sup>c</sup> PSC - manifold  $N$ . Then  $L$  is a PSC - manifold.*

*Proof.* We consider a Riemannian metric on  $L$  such that  $p : L \rightarrow M$  is a Riemannian submersion with totally geodesic fibers (see [AB, 9.60]). Then the result follows from O'Neil formulas (see [AB, 9.70]) applied to the Riemannian metric possibly conformally changed along fibers.  $\square$

Tomei [T] proved that the set of  $(n+1) \times (n+1)$  real matrices  $(a_{ij})$  with  $a_{ij} = 0$  whenever  $|i-j| > 1$  with fixed distinct all real eigenvalues is an  $n$ -manifold  $W_n$ . Using the Coxeter group technique from [D1], Tomei proved that  $W_n$  is aspherical. Then Davis [D2] proved among other things the following:

**2.4. Theorem** (Davis). *Tomei manifolds  $W_n$  are stably parallelizable.*

We recall that a manifold is stably parallelizable if its tangent bundle becomes parallelizable after adding a trivial bundle.

Recently Gaifullin used Tomei manifolds in the following Realization Theorem [Ga].

**2.5. Theorem** (Gaifullin). *For every integral homology class  $a \in H_n(X)$  there is a finite covering  $p; W'_n \rightarrow W_n$  of the Tomei manifold and a map  $f : W'_n \rightarrow X$  with  $f_*([W'_n]) = a$ .*

In the paper we use basic notations and facts from the surgery theory and bordism theory. In particular, we need the following:

**2.6. Theorem** (Surgery below the middle dimension [Wa]). *Let  $M$  be a stably parallelizable smooth  $n$ -manifold and let  $S \subset M$  be a smoothly embedded  $k$ -sphere with  $k < n/2$ . Then on  $S$  one can perform a surgery with the resulting manifold still stably parallelizable.*

We call a manifold  $r$ -stably parallelizable if the restriction of the tangent bundle to the  $r$ -skeleton  $M^{(r)}$  is stably parallelizable. We note that in the above theorem one can replace a stably parallelizable manifold by  $r$ -stably parallelizable if the surgery is performed on spheres of dimension  $\leq r$ . Then the resulting manifold will be  $r$ -stably parallelizable. Also we note that the spin manifolds can be characterized as those which are 2-stably parallelizable.

### 3. INESSENTIAL MANIFOLDS

We recall the following Gromov's definition [G3]:

**3.1. Definition.** An  $n$ -manifold  $M$  with the fundamental group  $\pi$  is called *essential* if its classifying map  $u^M : M \rightarrow B\pi$  cannot be deformed into the  $(n-1)$ -skeleton  $B\pi^{(n-1)}$  and it is called *inessential* if  $u^M$  can be deformed into  $B\pi^{(n-1)}$ .

Note that for an inessential  $n$ -manifold  $M$  we have  $\dim_{mc} \widetilde{M} \leq n-1$ . Indeed, a lift  $\widetilde{u}^M : \widetilde{M} \rightarrow E\pi^{(n-1)}$  of a classifying map is a uniformly cobounded map to an  $(n-1)$ -complex. Generally, if a classifying map  $u^M : M \rightarrow B\pi$  can be deformed to the  $k$ -dimensional skeleton, then  $\dim_{mc} \widetilde{M} \leq k$ .

Thus, one can consider a stronger version of Gromov's Conjecture:

**3.2. Conjecture** (The Strong Gromov Conjecture). *A classifying map  $u^M : M \rightarrow B\pi$  of the universal covering  $\widetilde{M}$  of a closed PSC  $n$ -manifold  $M$  can be deformed to the  $(n-2)$ -dimensional skeleton.*

Since this conjecture is false for finite cyclic groups, it requires some restrictions on  $\pi$ , like to be torsion free. A virtual version of it seems more reasonable:

**3.3. Conjecture.** *A closed manifold with a positive scalar curvature is virtually inessential, i.e., it admits a finite covering which is inessential.*

Clearly, this conjecture implies the Weak Gromov Conjecture.

We note that in [BD] we proved the Strong Gromov's Conjecture in the spin case under the RS-conditions, whereas in the almost spin case [Dr1] only the original version of the Gromov's conjecture was proved under similar conditions on  $\pi$ .

The inessentiality of a manifold can be characterized as follows [Ba] (see also [BD], Proposition 3.2).

**3.4. Theorem.** *Let  $M$  be a closed oriented  $n$ -manifold. Then the following are equivalent:*

1.  *$M$  is inessential;*
2.  *$u_*^M([M]) = 0$  in  $H_n(B\pi)$  where  $[M]$  is the fundamental class of  $M$ .*

In [BD] we proved the following addendum to Theorem 3.4.

**3.5. Proposition** ([BD], Lemma 3.5). *For an inessential manifold  $M$  with a CW complex structure a classifying map  $u : M \rightarrow B\pi$  can be chosen such that*

$$u(M^{(n-1)}) \subset B\pi^{(n-2)}.$$

Theorem 3.4 leads to the following

**3.6. Definition** ([G1]). A closed orientable  $n$ -manifold  $M$  is called rationally inessential if  $u_*^M([M]) = 0$  in  $H_n(B\pi; \mathbb{Q})$ .

The following theorem was proven by Rosenberg.

**3.7. Theorem** ([R1]). *If the fundamental group  $\pi$  of an almost spin PSC manifold  $M$  satisfies the Strong Novikov Conjecture, then  $M$  is rationally inessential.*

There is an analog of Theorem 3.4 for the universal coverings.

**3.8. Theorem** ([Dr1]). *Let  $M$  be a closed oriented  $n$ -manifold and let  $\tilde{u} : \tilde{M} \rightarrow E\pi$  be a lift of  $u^M$ . Then the following are equivalent:*

1.  *$\dim_{mc} \tilde{M} \leq n - 1$ ;*
2.  *$\tilde{u}_*([\tilde{M}]) = 0$  in  $H_n^{lf}(E\pi)$  where  $[\tilde{M}]$  is the fundamental class.*

We recall our main result from [BD].

**3.9. Theorem.** *Let  $M$  be an inessential closed spin PSC  $n$ -manifold with a fundamental group satisfying the RS-conditions. Then a classifying map  $u^M : M \rightarrow B\pi$  can be deformed into  $B\pi^{(n-2)}$ .*

*In particular,  $\dim_{mc} \tilde{M} \leq n - 2$ .*

We use the standard notation  $\pi_*^s$  and  $ko_*$  for the stable homotopy and for the connective real  $K$ -theory.

The mapping cone of a map  $S^k \rightarrow S^k$  of degree  $m$  is called the Moore space  $M(\mathbb{Z}_m, k)$ .

**3.10. Proposition.** *The natural transformation  $\pi_*^s(pt) \rightarrow ko_*(pt)$  induces an isomorphism  $\pi_n^s(X) \rightarrow ko_n(X)$  in any of the following cases:  $X = S^{n-1}$ ,  $X = S^{n-2}$ , and  $X$  is the Moore space  $M(\mathbb{Z}_m, n-2)$ .*

*Proof.* The first two cases follow from the isomorphisms

$$\pi_i^s(S^0) = \mathbb{Z}_2 \rightarrow ko_i(S^0) = \mathbb{Z}_2$$

for  $i = 1, 2$ . The third case follows from the Five Lemma applied to the cofibration  $S^{n-2} \rightarrow S^{n-2} \rightarrow M(\mathbb{Z}_m, n-2)$ .  $\square$

**3.11. Corollary.** *For any CW complex  $K$  the natural homomorphism  $\pi_n^s(K^{(n-1)}/K^{(n-3)}) \rightarrow ko_n(K^{(n-1)}/K^{(n-3)})$  is an isomorphism.*

*Proof.* By the Minimal Cell Structure Theorem (see [BD, Prop. 2.1]  $K^{(n-1)}/K^{(n-3)}$  is homotopy equivalent to the wedge of spheres of dimensions  $n-1$  and  $n-2$  and the Moore spaces  $M(\mathbb{Z}_m, n-2)$ .  $\square$

**3.12. Lemma.** *For any CW complex  $K$  the homomorphism*

$$g_* : \pi_n^s(K/K^{(n-3)}) \rightarrow ko_n(K/K^{(n-3)})$$

*induced by the transformation of spectra  $g : \pi^s \rightarrow ko$  is an isomorphism.*

*Proof.* Since  $\pi^s$  and  $ko$  are connective spectra it suffices to prove that the homomorphism

$$g_* : \pi_n^s(K^{(n+1)}/K^{(n-3)}) \rightarrow ko_n(K^{(n+1)}/K^{(n-3)})$$

is an isomorphism. Let  $A = K^{(n)}/K^{(n-3)}$  and  $B = K^{(n-1)}/K^{(n-3)}$ . We consider the commutative diagram generated by the exact sequences of the pair  $(A, B)$ :

$$\begin{array}{ccccccccc} \oplus \mathbb{Z}_2 & \longrightarrow & \pi_n^s(B) & \longrightarrow & \pi_n^s(A) & \longrightarrow & \oplus \mathbb{Z} & \longrightarrow & \pi_{n-1}^s(B) \\ \downarrow \cong & & \downarrow h_*^1 & & \downarrow h_* & & \downarrow \cong & & \downarrow h_*^2 \\ \oplus \mathbb{Z}_2 & \longrightarrow & ko_n(B) & \longrightarrow & ko_n(A) & \longrightarrow & \oplus \mathbb{Z} & \longrightarrow & ko_{n-1}(B). \end{array}$$

By Corollary 3.11 the homomorphism  $h_*^1$  is an isomorphism. It was proven in [BD, Prop.2.2]) that  $h_*^2$  is an isomorphisms. From the Five Lemma we obtain that  $h_*$  is an isomorphism.

Now the lemma follows from the commutative diagram generated by the exact sequences of the pair  $(K^{(n+1)}/K^{(n-3)}, K^{(n)}/K^{(n-3)})$  and the Five Lemma:

$$\begin{array}{ccccccc}
\oplus \mathbb{Z} & \longrightarrow & \pi_n^s(K^{(n)}/K^{(n-3)}) & \longrightarrow & \pi_n^s(K^{(n+1)}/K^{(n-3)}) & \longrightarrow & 0 \\
\text{iso} \downarrow & & h_* \downarrow & & g_* \downarrow & & \text{iso} \downarrow \\
\oplus \mathbb{Z} & \longrightarrow & ko_n(K^{(n)}/K^{(n-3)}) & \longrightarrow & ko_n(K^{(n+1)}/K^{(n-3)}) & \longrightarrow & 0.
\end{array}$$

□

We need the following:

**3.13. Lemma** ([BD], Lemma 4.1). *Suppose that a classifying map  $f : M \rightarrow B\pi$  of a closed spin  $n$ -manifold,  $n \geq 4$  satisfies  $f_*([M]_{ko}) = 0$ . Then  $f$  is homotopic to a map  $g : M \rightarrow B\pi^{(n-2)}$  that agrees with  $f$  on  $M^{(n-2)}$ .*

Note that for PSC manifolds the RS-conditions together with Rosenberg's theorem imply the equality  $f_*([M]_{ko}) = 0$ .

In the case when the second homotopy group of a manifold is trivial we can strengthen our main result from [BD] to the following.

**3.14. Theorem.** *Let  $M$  be an inessential closed spin PSC  $n$ -manifold with a fundamental group satisfying the RS-conditions and with  $\pi_2(M) = 0$ . Then a classifying map  $u^M : M \rightarrow B\pi$  can be deformed into  $B\pi^{(n-3)}$ . In particular,  $\dim_{mc} \widetilde{M} \leq n - 3$ .*

*Proof.* We fix a CW complex structure on  $M$  with one top dimensional cell. To make notations shorter we set  $f = u^M$ . Using the main result of [BD] (Theorem 3.9) we may assume that  $f$  is cellular map and  $f(M) \subset B\pi^{(n-2)}$ . Let  $o_{n-2}(f) \in H^{n-2}(M; \pi_{n-3}(F))$  be the primary obstruction to deform the restriction  $f|_{M^{(n-2)}}$  to  $B\pi^{(n-3)}$  or, equivalently, to lift  $f : M \rightarrow B$  to the total space  $E$  of the fibration  $F \rightarrow E \rightarrow B$  corresponding to the inclusion  $B\pi^{(n-3)} \rightarrow B\pi^{(n-2)}$ . Since  $\pi_{n-3}(F) \cong \pi_{n-2}(B\pi^{(n-2)}, B\pi^{(n-3)})$  is a free  $\mathbb{Z}\pi$ -module we obtain [Br]

$$H^{n-2}(M; \pi_{n-3}(F)) \cong H_c^{n-2}(\widetilde{M}; \oplus \mathbb{Z}) \cong H_2(\widetilde{M}; \oplus \mathbb{Z}) \cong \oplus \pi_2(M) = 0.$$

Hence,  $o_{n-2}(f) = 0$ . Let  $f^{(n-2)} : M^{(n-2)} \rightarrow B\pi^{(n-3)}$  be the result of a deformation of  $f|_{M^{(n-2)}}$  to  $B\pi^{(n-3)}$ . Since  $E\pi^{(i)}$  is contractible in  $E\pi^{(i+1)}$ , we can extend the map  $f^{(n-2)}$  to a map

$$g^{(n-1)} : M^{(n-1)} \rightarrow B\pi^{(n-2)}.$$

Clearly, we can extend  $g^{(n-1)}$  to a classifying map  $g : M \rightarrow B\pi$ . In view of the RS-conditions we can apply to  $g$  Lemma 3.13. Hence, we may assume that  $g(M) \subset B\pi^{(n-2)}$  and  $g|_{M^{(n-2)}} = f^{(n-2)}$ .



Let  $o_{n-1}(g) \in H^{n-1}(M; \pi_{n-2}(F))$  be the primary obstruction to deform  $g|_{M^{(n-1)}}$  to  $B\pi^{(n-3)}$ . Since  $\pi_{n-2}(F) \cong \pi_{n-1}(B\pi^{(n-2)}, B\pi^{(n-3)})$  is a free  $\mathbb{Z}_2\pi$ -module, we obtain

$$H^{n-1}(M; \pi_{n-2}(F)) \cong H_c^{n-1}(\widetilde{M}; \oplus \mathbb{Z}_2) \cong H_1(\widetilde{M}; \oplus \mathbb{Z}_2) = 0.$$

Thus,  $o_{n-1}(g) = 0$ . Let  $f^{(n-1)} : M^{(n-1)} \rightarrow B\pi^{(n-3)}$  be the result of that deformation. We note that any extension  $f' : M^n \rightarrow B\pi$  of  $f^{(n-1)}$  to  $M^n$  is a classifying map for  $M$ . The primary obstruction  $o_{f'}$  to extend  $f^{(n-1)}$  to  $f^{(n)} : M^n \rightarrow B\pi^{(n-3)}$  lies in the group

$$H^n(M; \pi_n(B\pi, B\pi^{(n-3)})) = H_0(M; \pi_n(B\pi, B\pi^{(n-3)})) = \pi_n(B\pi, B\pi^{(n-3)})_\pi.$$

We note that for the group of coinvariants there is the equality

$$\pi_n(B\pi, B\pi^{(n-3)})_\pi = \pi_n(B\pi/B\pi^{(n-3)}).$$

Also note that the obstruction class  $o_{f'}$  in  $\pi_n(B\pi/B\pi^{(n-3)})$  is represented by  $\bar{f}_*(1)$  for the homomorphism

$$\bar{f}_* : \pi_n(M/M^{(n-1)}) = \pi_n(S^n) \rightarrow B\pi/B\pi^{(n-3)}$$

where  $\bar{f} : M/M^{(n-1)} = S^n \rightarrow B\pi/B\pi^{(n-3)}$  is induced by the map  $f'$ .

We argue that  $o_{f'} = 0$ . Assume not,  $o_{f'} \neq 0$ . Consider the commutative diagram generated by  $\bar{f}$ :

$$\begin{array}{ccc} \pi_n(S^n) & \xrightarrow{\bar{f}_*} & \pi_n(B\pi/B\pi^{(n-3)}) \\ \cong \downarrow & & \cong \downarrow \\ \pi_n^s(S^n) & \xrightarrow{\bar{f}_*} & \pi_n^s(B\pi/B\pi^{(n-3)}) \\ \cong \downarrow & & \cong \downarrow \\ ko_n(S^n) & \xrightarrow{\bar{f}_*} & ko_n(B\pi/B\pi^{(n-3)}). \end{array}$$

The vertical low right arrow is an isomorphism by Lemma 3.12. By the definition the image of the fundamental class  $[M^n]_{ko}$  after factorization  $M^n \rightarrow M^n/M^{(n-1)}$  is a generator of  $ko_n(M^n/M^{(n-1)}) = ko_n(S^n) = \mathbb{Z}$ . The commutative diagram

$$\begin{array}{ccc} ko_n(M^n) & \xrightarrow{f'_*} & ko_n(B\pi) \\ \downarrow & & \downarrow \\ ko_n(S^n) & \xrightarrow{\bar{f}_*} & ko_n(B\pi/B\pi^{(n-3)}) \end{array}$$

implies  $f'_*([M^n]_{ko}) = \bar{f}_*([M^n]_{ko}) \neq 0$ . We obtain a contradiction with Rosenberg's theorem and the RS-conditions.  $\square$

## 4. FREE LOOP SPACES

By  $L_0X$  we denote the space of free null-homotopic loops on  $X$ ,  $L_0X \subset \text{Map}(S^1, X)$ . When  $X$  is simply connected  $L_0X = LX$ , the free loop space. The group  $S^1$  naturally acts on  $L_0X$  by  $(zf)(u) = f(z^{-1}u)$ . Let  $S_0(X)$  denote the orbit space  $L_0X/S^1$ , the space of null-homotopic 'strings' on  $X$ .

Suppose that  $X$  is a proper metric space which uniformly locally path connected, i.e. there are  $\epsilon > 0$  and a continuous map  $\Phi : W_\epsilon \times [0, 1] \rightarrow X$  of the  $\epsilon$ -neighborhood  $W_\epsilon$  of the diagonal  $\Delta(X) \subset X \times X$  such that  $\Phi(x, y, -) : [0, 1] \rightarrow X$  is a path from  $x$  to  $y$  and  $\Phi(x, x, -)$  is a constant path for all  $(x, y) \in W_\epsilon$ . Then  $L_0X$  is uniformly locally path connected by means of the function  $\tilde{\Phi} : \widetilde{W}_\epsilon \times [0, 1] \rightarrow L_0X$  defined as  $\tilde{\Phi}(f_1, f_2, t)(u) = \Phi(f_1(u), f_2(u), t)$  where  $\widetilde{W}_\epsilon$  is the epsilon neighborhood of the diagonal  $\Delta(L_0X)$  in the uniform metric. We note that  $\widetilde{W}_\epsilon$  is invariant with respect to the diagonal  $S^1$ -action and the map  $\tilde{\Phi}$  is equivariant. Then the space  $S_0(X)$  is locally contractible. Moreover, the space  $S_0(X)/X$  is locally contractible. Note that for such spaces the singular cohomology behaves well and agrees with the Čech cohomology [Bre].

Note that  $X$  is embedded in  $LX$  as well in  $S_0X$  via the constant loops. Let  $j_X : X \rightarrow S_0(X)$  denote that embedding. Note that  $S_0$  is a covariant functor in the category of topological spaces and  $j_X : 1 \rightarrow S_0$  is a natural transformation. We consider the Borel construction for the  $S^1$  action on  $L_0X/X$  and on  $S^{2N+1}$  for sufficiently large  $N$ :

$$\begin{array}{ccccc} S^{2N+1} & \longleftarrow & S^{2N+1} \times (L_0X/X) & \longrightarrow & L_0X/X \\ \downarrow & & \downarrow & & \downarrow \\ \mathbb{C}P^N & \xleftarrow{p_1} & S^{2N+1} \times_{S^1} (L_0X/X) & \xrightarrow{p_2} & S_0(X)/X. \end{array}$$

Every orbit  $S^1z \in S_0(X)/X$  is homeomorphic to  $S^1/G_z$  where  $G_z \subset S^1$  is the stabilizer of  $z$ , a closed subgroup of  $S^1$ . Then the fiber  $p_2^{-1}(z)$  is homeomorphic to  $S^{2N+1}/G_z$ . We note that the special fiber  $F = p_2^{-1}(\{X\})$  of  $p_2$  is homeomorphic to  $\mathbb{C}P^N$ . We denote by

$$E(N, X) = (S^{2N+1} \times_{S^1} (L_0X/X))/F$$

and by  $\bar{p}_2 : E(N, X) \rightarrow S_0(X)/X$  the induced map. Thus  $p_2 = \bar{p}_2 \circ q$  where  $q$  is the map collapsing  $F$ .

**4.1. Proposition.** *For a uniformly locally path connected proper metric space  $X$  the projection  $\bar{p}_2$  induces an isomorphism*

$$(\bar{p}_2)_* : H_i(E(N, X); \mathbb{Q}) \rightarrow H_i(S_0(X)/X; \mathbb{Q})$$

in dimensions  $i \leq 2N$ .

*Proof.* Since the spaces here are locally nice, it suffices to prove the statement for the cohomology. We consider the Leray spectral sequence of  $\bar{p}_2$ . Since the map  $\bar{p}_2$  is proper, the sequence converges to  $H^*(E(N, X); \mathbb{Q})$  [Bre]. Note that  $\bar{p}_2$  has one exceptional fiber which is a point. All other fibers are lens spaces  $L_m^{2N+1}$  possibly degenerated to the sphere  $S^{2N+1}$ . Therefore, in the Leray spectral sequence we have trivial stalks

$$\mathcal{H}^q(\bar{p}_2) = \lim_{\rightarrow} H^q(\bar{p}_2^{-1}(U); \mathbb{Q}) = H^q(\bar{p}_2^{-1}(x); \mathbb{Q}) = 0$$

for  $0 < q \leq 2N$ . Hence,  $E_2^{p,q} = 0$  for  $0 < q \leq 2N$  and the result follows.  $\square$

The main result of this section, Theorem 4.4, is rather technical. It is a computation of the string homology in some special case. Since we are doing it by rather elementary means we don't involve here the loop homology and the string homology theories [CJY],[We].

**4.2. Proposition.** *Let  $X = B\pi/B\pi^{(n-1)}$ , for  $n > 3$ . Then  $H_i(LX/X; \mathbb{Q}) = 0$  for  $0 < i < n - 1$  and  $n - 1 < i < 2n - 3$ .*

*Proof.* Consider the evaluation fibration  $ev : LX \rightarrow X$  and note that it admits a section  $s : X \rightarrow LX$ . We claim that the rationalization of the fiber  $\Omega X$  has the rational homology groups of a wedge of  $(n - 1)$ -spheres.

The homotopy exact sequence of the pair  $(B\pi, B\pi^{(n-1)})$  implies that for  $i > 2$ ,

$$\pi_i(B\pi, B\pi^{(n-1)}) = \pi_{i-1}(B\pi^{(n-1)}) = \pi_{i-1}(E\pi^{(n-1)}) = \pi_{i-1}(\vee S^{n-1}).$$

Clearly,  $\pi_i(B\pi, B\pi^{(n-1)}) = 0$  for  $i < n$  and  $\pi_n(B\pi, B\pi^{(n-1)}) = \oplus \mathbb{Z}$ . Note that  $\pi_i(B\pi, B\pi^{(n-1)}) \otimes \mathbb{Q} = 0$  for  $i - 1 < 2n - 3$ . Since

$$\pi_i(B\pi/B\pi^{(n-1)}) = \pi_i(B\pi, B\pi^{(n-1)})_\pi$$

is the set of coinvariants of  $\pi_i(B\pi, B\pi^{(n-1)})$ , it follows that

$$\pi_i(B\pi/B\pi^{(n-1)}) \otimes \mathbb{Q} = 0$$

for  $n < i < 2n - 2$ . Clearly,  $\pi_i(B\pi/B\pi^{(n-1)}) = 0$  for  $i < n$ . Then the rationalization  $X_0$  of  $X$  has trivial homotopy groups in dimensions  $i < n$  and  $i > n$ . Then the rationalization of the loop space  $\Omega(X)_0 = \Omega(X_0)$  has trivial homotopy groups in dimensions  $i < n - 1$  and  $n - 1 < i < 2n - 3$ .

Then in the homology Leray-Serre spectral sequence of  $ev$  we have  $E_{p,q}^2 = 0$  for  $0 < q < n-1$  and  $n-1 < q < 2n-3$ . Since  $X$  is  $(n-1)$ -connected,  $E_{p,0}^2 = 0$  for  $0 < p < n$ . This implies that  $H_i(LX; \mathbb{Q}) = 0$  for  $0 < i < n-1$  and  $n < i < 2n-3$ . Since  $ev$  has a section, the inclusion  $X \rightarrow LX$  induces an isomorphism  $H_n(X; \mathbb{Q}) = H_n(LX; \mathbb{Q})$ . The splitting generated by the section

$$H_i(LX) = H_i(X) \oplus H_i(LX/X)$$

implies that  $H_i(LX/X; \mathbb{Q}) = 0$  for  $0 < i < n-1$  and  $n-1 < i < 2n-3$ .  $\square$

**4.3. Proposition.** *Let  $\phi_{*,*} : E_{*,*}(p') \rightarrow E_{*,*}(p)$  be the morphism of the Leray-Serre spectral sequences generated by a morphism  $(f', f) : p' \rightarrow p$  of fiber bundles,  $p' : X' \rightarrow Y'$  and  $p : X \rightarrow Y$ .*

(a) *If  $\phi_{0,n}^2 : E_{0,n}^2(p') \rightarrow E_{0,n}^2(p)$  is surjective then also is  $\phi_{0,n}^\infty : E_{0,n}^\infty(p') \rightarrow E_{0,n}^\infty(p)$ .*

(b) *If all  $\phi_{k,l}^\infty : E_{k,l}^\infty(p') \rightarrow E_{k,l}^\infty(p)$  for  $k+l = n$  are surjective, then  $f'_* : H_n(X') \rightarrow H_n(X)$  is surjective.*

*Proof.* (a) We show recursively that  $\phi_{0,n}^r$  is surjective. Since in the commutative diagram

$$\begin{array}{ccccccc} \xrightarrow{d^r} & E_{0,n}^r(p') & \longrightarrow & E_{0,n}^{r+1}(p') & \longrightarrow & 0 \\ & \phi_{0,n}^r \downarrow & & \phi_{0,n}^{r+1} \downarrow & & \\ \xrightarrow{d^r} & E_{0,n}^r(p) & \xrightarrow{\psi} & E_{0,n}^{r+1}(p) & \longrightarrow & 0 \end{array}$$

$\phi_{0,n}^r$  and  $\psi$  are surjective,  $\phi_{0,n}^{r+1}$  is surjective.

(b) The homology groups  $H_n(X)$  is obtained from  $E_{n,0}^\infty(p)$  by consecutive extensions by  $E_{n-1,1}^\infty(p)$  then by  $E_{n-2,2}^\infty(p)$  and so on up to  $E_{0,n}^\infty$ . The same holds true for  $H_n(X')$ . Let  $A_0 = E_{n,0}^\infty(p)$ ,  $A'_0 = E_{n,0}^\infty(p')$  and let  $A_i$  and  $A'_i$  denote the corresponding intermediate extensions,  $i = 1, \dots, n$ . Thus,  $H_n(X) = A_n$  and  $H_n(X') = A'_n$ . We apply the epimorphism version of Five Lemma recursively to the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & E_{n-i,i}^\infty(p') & \longrightarrow & A'_i & \longrightarrow & A'_{i-1} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & E_{n-i,i}^\infty(p) & \longrightarrow & A_i & \longrightarrow & A_{i-1} & \longrightarrow & 0 \end{array}$$

to obtain the result.  $\square$

**4.4. Theorem.** *Suppose that the classifying space  $B\pi$  of a discrete group  $\pi$  is locally finite. Then the inclusion*

(a)  $j_\pi : B\pi \rightarrow S_0(B\pi)$  induces an isomorphism

$$(j_\pi)_* : H_n(B\pi; \mathbb{Q}) \rightarrow H_n(S_0(B\pi); \mathbb{Q})$$

of the rational homology groups and

(b) for  $n \geq 5$  the inclusion homomorphism

$$H_n(S_0(B\pi^{(n-2)}); \mathbb{Q}) \rightarrow H_n(S_0(B\pi); \mathbb{Q})$$

is zero.

*Proof.* (a) We show that  $H_*(S_0(B\pi), j_\pi(B\pi); \mathbb{Q}) = 0$ .

Note that there is a fibration  $p : L(B\pi) \rightarrow B\pi$  with the fiber  $\Omega(B\pi)$  which is homotopy equivalent to  $\pi$ . This fibration admits a section  $s$  and the path component of  $s(B\pi)$  is exactly  $L_0(B\pi)$ . Thus, the restriction  $p_0$  of  $p$  to  $L_0(B\pi)$  is a fibration with homotopy trivial fiber. Hence the inclusion  $B\pi \rightarrow L_0(B\pi)$  is a homotopy equivalence. Thus,  $L_0(B\pi)/B\pi$  is contractible. Therefore the projection in the Borel construction

$$p_1 : S^{2N+1} \times_{S^1} (L_0(B\pi)/B\pi) \rightarrow \mathbb{C}P^N$$

induces isomorphism of homology groups. By Proposition 4.1 the projection

$$\bar{p}_2 : (S^{2N+1} \times_{S^1} (L_0(B\pi)/B\pi))/F \rightarrow S_0(B\pi)/j_\pi(B\pi)$$

induces isomorphisms of rational homology in dimensions  $\leq 2N$ . The exceptional fiber  $F$  of  $p_2$  is homeomorphic to  $\mathbb{C}P^N$ . It defines a section of  $p_1$ . Thus collapsing  $F$  to a point defines a contractible space. Therefore,  $H_i(S_0(B\pi), j_\pi(B\pi); \mathbb{Q}) = 0$  for  $i \leq 2N$ . Since  $N$  is arbitrary, the result follows.

(b) In view of the exact sequence of the pair  $(S_0(B\pi), S_0(B\pi^{(n-2)}))$  it suffices to show that the homomorphism

$$q_* : H_n(S_0(B\pi); \mathbb{Q}) \rightarrow H_n(S_0(B\pi)/S_0(B\pi^{(n-2)}); \mathbb{Q})$$

generated by the collapsing map

$$q : S_0(B\pi) \rightarrow S_0(B\pi)/S_0(B\pi^{(n-2)})$$

is injective. We note that the collapsing map  $\psi : B\pi \rightarrow B\pi/B\pi^{(n-2)}$  induces an equivariant map  $L_0(\psi) : L_0(B\pi) \rightarrow L_0(B\pi/B\pi^{(n-2)})$  and, hence, it defines a map  $s(\psi) : S_0(B\pi) \rightarrow S_0(B\pi/B\pi^{(n-2)})$ . Note that the map  $s(\psi)$  factors through  $q$ ,  $s(\psi) = \psi' \circ q$ . Thus, it suffices to

show that  $s(\psi)$  is injective for  $n$ -homology. Since  $(j_\pi)_*$  and  $i$  are isomorphisms in the commutative diagram

$$\begin{array}{ccc} H_n(B\pi/B\pi^{(n-2)}; \mathbb{Q}) & \xrightarrow{j'_*} & H_n(S_0(B\pi/B\pi^{(n-2)}); \mathbb{Q}) \\ i \uparrow & & s(\psi) \uparrow \\ H_n(B\pi; \mathbb{Q}) & \xrightarrow{(j_\pi)_*} & H_n(S_0(B\pi); \mathbb{Q}), \end{array}$$

it suffices to show that  $j'_*$  is a monomorphism. Let  $X = B\pi/B\pi^{(n-2)}$ . We need to show that

$$j'_* : H_n(X; \mathbb{Q}) \rightarrow H_n(S_0X; \mathbb{Q})$$

is a monomorphism. The exact sequence of the pair  $(S_0(X), X)$  reduces the problem to showing that the collapsing map  $q : S_0(X) \rightarrow S_0(X)/X$  induces an epimorphism  $q_* : H_{n+1}(S_0(X); \mathbb{Q}) \rightarrow H_{n+1}(S_0(X)/X; \mathbb{Q})$ .

We consider the diagram generated by Borel's constructions for  $S^1$  action on  $LX$ ,  $L'/X$  and  $S^{2N+1}$  with  $N > n$ .

$$\begin{array}{ccc} H_{n+1}(S^{2N+1} \times_{S^1} LX; \mathbb{Q}) & \xrightarrow{(p'_2)_*} & H_{n+1}(S_0X; \mathbb{Q}) \\ \bar{q}_* \downarrow & & q_* \downarrow \\ H_{n+1}(S^{2N+1} \times_{S^1} (LX/X); \mathbb{Q}) & \xrightarrow{(p_2)_*} & H_{n+1}((S_0X)/X; \mathbb{Q}). \end{array}$$

It suffices to show that both  $(p_2)_*$  and  $\bar{q}_*$  are surjective. By Proposition 4.1 the reduced projection

$$\bar{p}_2 : (S^{2N+1} \times_{S^1} (LX/X))/p_2^{-1}(*) \rightarrow S_0X/X$$

induces isomorphism of the homology in dimensions  $\leq 2N$ . Here  $*$  =  $\{X\}$  is the orbit of the fixed point. The projection

$$p_1 : S^{2N+1} \times_{S^1} (LX/X) \rightarrow \mathbb{C}P^N$$

is a locally trivial bundle with the fiber  $LX/X$ . As before, the fixed point  $*$  of the action defines a section  $s$  of  $p_1$ . Therefore there is a splitting

$$H_i(S^{2N+1} \times_{S^1} (LX/X)) = H_i(\mathbb{C}P^N) \oplus H_i((S^{2N+1} \times_{S^1} (LX/X))/p_2^{-1}(*))$$

generated by  $p_1$  and the collapsing map. Hence  $(p_2)_*$  is surjective.

Similarly any fixed point of the  $S^1$  action on  $LX$  defines a section of  $p'_1 : S^{2N+1} \times_{S^1} (LX) \rightarrow \mathbb{C}P^N$ . We consider the morphism of the Leray-Serre spectral sequence  $\pi_{*,*} : E_{*,*}(p'_1) \rightarrow E_{*,*}(p_1)$  of  $p_1$  and  $p'_1$  generated by  $\bar{q}$ .

By Proposition 4.2,  $H_i(LX/X; \mathbb{Q}) = 0$  for  $i < n - 2$  and  $n - 2 < i < 2n - 5$ . Note that since in the definition of  $X$  we collapse here

the  $(n - 2)$ -skeleton of  $B\pi$  instead of the  $(n - 1)$ -skeleton, the inequalities for  $i$  are shifted by one. The assumption  $n \geq 5$  implies that  $H_{n-1}(LX/X; \mathbb{Q}) = 0$  and hence,  $E_{2,n-1}^2(p_1) = 0$ . Then it is easy to see that nonzero elements on the  $(n + 1)$ -diagonal are only  $E_{0,n+1}^2$  and  $E_{n+1,0}^2$ . Since  $H_{n+1}(LX; \mathbb{Q}) \rightarrow H_{n+1}(LX/X; \mathbb{Q})$  is surjective,  $\phi_{0,n+1}^2 : E_{0,n}^2(p'_1) \rightarrow E_{0,n}^2(p_1)$  is surjective. Clearly,  $\phi_{n+1,0}^2$  is an isomorphism:  $E_{n+1,0}^2(p_1) = E_{n+1,0}^2(p'_1) = H_{n+1}(\mathbb{C}P^m; \mathbb{Q})$ . By Proposition 4.3 part (a),  $\phi_{0,n+1}^\infty$  is surjective. Since both fibrations have sections, we have stabilizations:  $E_{n+1,0}^2(p_1) = E_{n+1,0}^\infty(p_1)$  and  $E_{n+1,0}^2(p'_1) = E_{n+1,0}^\infty(p'_1)$ . Then by Proposition 4.3 part (b),  $\bar{q}_*$  is surjective in dimension  $n + 1$ .  $\square$

## 5. WEAK GROMOV CONJECTURE

We need the following refinement of the Realization Theorem.

**5.1. Lemma.** *For every  $a \in H_n(B\pi)$ ,  $n \geq 5$ , there is a stably parallelizable closed oriented  $n$ -manifold and a map  $f : N \rightarrow B\pi$  with  $f_*([N]) = ka$  for some  $k \in \mathbb{N}$  which induces an isomorphism of the fundamental groups.*

*Proof.* By Gaifullin's Realization Theorem (Theorem 2.5), there is a finite covering  $p : W' \rightarrow W$  of the Tomei manifold  $W$  and a map  $f' : W' \rightarrow B\pi$  with  $f'_*([W']) = ka$  for some  $k \in \mathbb{N}$ . By Davis' Theorem (Theorem 2.4),  $W$  is stably parallelizable. Then so is  $W'$ .

Let  $\{\phi_i : S^1 \rightarrow B\pi\}_{i \in J}$  be a finite set of loops generating  $\pi$ . We consider the connected sum  $L = \#_{i \in J}(S^1 \times S^{n-1}) \# W'$  and note that it is stably parallelizable. Then we define a map  $f_0 : L \rightarrow B\pi$  which induces an epimorphism of the fundamental groups. We define it as the composition  $(g \vee f') \circ q$  where the map  $q : L \rightarrow \vee_{i \in J}(S^1 \times S^{n-1}) \vee W'$  is defined as follows. We may assume that all manifolds participating in the above connecting sum are connected to an  $n$ -sphere with a set of holes. Then the map  $q$  is defined by collapsing the sphere with holes to a point. The map  $g : \vee_{i \in J}(S^1 \times S^{n-1}) \rightarrow B\pi$  is the composition  $g = \vee \phi_i \circ p'$ . Here  $p' : \vee_{i \in J}(S^1 \times S^{n-1}) \rightarrow \vee_{i \in J} S^1$  is the wedge of the projections  $S^1 \times S^{n-1} \rightarrow S^1$  onto the first factor. We perform a finite sequence of 1-surgeries on  $L$  to turn  $f_0$  to a map  $f : N \rightarrow B\pi$  that induces an isomorphism of the fundamental group. In view of Theorem 2.6 we may assume that  $N$  is stably parallelizable.

Clearly,  $f_*([N]) = f'_*([W'])$ .  $\square$

Let  $\xi : S^\infty \rightarrow \mathbb{C}P^\infty$  denote the universal  $S^1$ -bundle.

**5.2. Proposition.** *Suppose that a manifold  $N$  has an integral Stiefel-Whitney class  $w_2$  which comes from the fundamental cohomology class under a map  $g : N \rightarrow \mathbb{C}P^\infty$  that induces an isomorphism of the 2-dimensional homotopy groups. Suppose that  $p : L \rightarrow N$  is the pull-back of the bundle  $\xi$  by a map  $g$ . Then  $\pi_2(L) = 0$ , the map  $p$  induces an isomorphism of the fundamental groups  $p_* : \pi_1(L) \rightarrow \pi_1(N)$ , and  $L$  is a spin manifold.*

*Proof.* The Five Lemma applied to the commutative diagram generated by the homotopy exact sequences of  $\xi$  and  $p$

$$\begin{array}{ccccccc} 0 & \longrightarrow & \pi_2(L) & \longrightarrow & \pi_2(N) & \xrightarrow{\partial} & \pi_1(S^1) \\ & & \downarrow & & \cong \downarrow g_* & & id \downarrow \\ 0 & \longrightarrow & \pi_2(S^\infty) = 0 & \longrightarrow & \pi_2(\mathbb{C}P^\infty) & \xrightarrow{\cong} & \pi_1(S^1) \end{array}$$

implies that  $\pi_2(L) = 0$ . Since  $\partial$  is an isomorphism, the homotopy exact sequence of  $p$

$$0 \rightarrow \pi_2(N) \xrightarrow{\partial} \pi_1(S^1) \rightarrow \pi_1(L) \xrightarrow{p_*} \pi_1(N) \rightarrow 0$$

implies that  $p_* : \pi_1(L) \rightarrow \pi_1(N)$  is an isomorphism.

We identify  $N$  with the zero section in the induced complex line bundle  $\nu : E \rightarrow N$ . Let  $D \rightarrow N$  be the corresponding unit disk bundle. Then  $L = \partial D$ . Note that for the tangent bundle we have  $TE = \nu^*(TN \oplus \nu)$ . By the definition of  $\nu$  the Stiefel-Whitney class  $w_2(\nu) = w_2(TN)$ . Since  $w_1(TN) = 0$ , we obtain

$$w_2(TE) = \nu^*(w_2(TN \oplus \nu)) = \nu^*(w_2(TN) + w_2(\nu) + w_1(TN)w_1(\nu)) = 0.$$

Thus,  $E$  and  $D$  are spin manifolds and, hence, so is  $L$ .  $\square$

We will refer to the groups satisfying the RS-conditions as to *RS-groups*. Here our main result. It can be considered as a non-spin analog of Theorem 3.7.

**5.3. Theorem.** *Let  $M$  be a non-spin closed orientable  $n$ -manifold,  $n \geq 5$ , with positive scalar curvature whose fundamental group  $\pi$  is an RS-group. Then  $M$  is rationally inessential.*

*Proof.* Let  $K$  be from Lemma 5.1 applied to  $u_*^M([M])$ . Thus,  $K$  is a spin manifold with a map  $f : K \rightarrow B\pi$  that induces isomorphism of the fundamental groups and with  $f_*([K]) = k(u_*^M([M]))$  for some  $k \in \mathbb{N}$ . Applying the surgery on generators of  $\pi_2(K)$  we may assume that  $\pi_2(K) = 0$ . We define  $C = \mathbb{C}P^2 \times S^{n-4}$  for  $n > 6$ . For  $n = 5, 6$  we define  $C$  to be the result of a  $(n - 4)$ -surgery on  $\mathbb{C}P^2 \times S^{n-4}$



performed on the factor  $S^{n-4}$ . Thus,  $C$  is a simply connected non-spin  $spin^c$ -manifold. Also we note that the connected sum  $N' = K \# C$  is totally non-spin with  $u_*^{N'}([N']) = f_*([K])$  and with  $\pi_2(N') = \mathbb{Z}\pi$ . Since  $K$  is parallelizable,  $N'$  is a  $spin^c$  manifold. Let  $g' : N' \rightarrow \mathbb{C}P^\infty$  be a map representing the integral 2nd Stiefel-Whitney class. We may assume that  $g'$  induces an epimorphism of the 2nd homotopy groups  $g'_* : \pi_2(N') \rightarrow \pi_2(\mathbb{C}P^\infty)$ . This is obvious for  $n \geq 7$ . For  $n = 5, 6$  we leave the proof to the reader.

Note that  $\ker(g')_* \subset \ker(\nu_{N'})$  where  $\nu_{N'} : N' \rightarrow BSO$  is the classifying map of the stable normal bundle on  $N'$ . Also note that  $\ker(g')_* = I(\pi)$  is a finitely generated  $\pi$ -module. By surgery in dimension 2 we can kill the kernel  $\ker(g')_*$  to obtain a  $spin^c$  totally non-spin manifold  $N$  with  $\pi_2(N) = \mathbb{Z}$  and with the map representing the integral 2nd Stiefel-Whitney class  $g : N \rightarrow \mathbb{C}P^\infty$  that induces an isomorphism  $g_* : \pi_2(N) \rightarrow \pi_2(\mathbb{C}P^\infty)$ .

Thus, a totally non-spin manifold  $N$  realizes a homology class of a PSC manifold  $\coprod_k M$  in  $H_*(B\pi)$ . Then by the Jung-Stolz theorem (Theorem 2.2)  $N$  admits a metric of positive scalar curvature.

We consider the pull-back manifold  $L$ ,  $p : L \rightarrow N$  with respect to  $g$  and  $\xi$ . By Theorem 2.3  $L$  is a PSC manifold. By Proposition 5.2  $L$  is spin with  $p_* : \pi_1(L) \rightarrow \pi_1(N)$  an isomorphism. By the assumption the fundamental group  $\pi_1(L) = \pi$  satisfies the RS conditions. Since  $\bar{f} \circ p : L \rightarrow B\pi^{(n)}$  induces an isomorphism of the fundamental groups, the  $(n+1)$ -manifold  $L$  is inessential. By Theorem 3.14 the map  $\bar{f} \circ p : L \rightarrow B\pi$  is deformable to  $B\pi^{(n-2)}$ . Let  $q : L \rightarrow B\pi^{(n-2)}$  be the result of such deformation.

The composition  $q \circ p^{-1}$  defines a continuous map  $\phi : N \rightarrow S_0(B\pi^{(n-2)})$  to the string space as follows. We fix a discontinuous section  $s : N \rightarrow L$ . For  $x \in N$  we define  $f_x : S^1 \rightarrow B\pi^{(n-2)}$  by the formula  $f_x(z) = q(z(s(x)))$  where  $z \in S^1$  acts on  $L$ . Then we define  $\phi(x) = S^1(f_x)$ , the orbit of  $f_x$ . Note that it does not depend on the choice of section. Note that  $\phi$  is homotopic in  $S_0(B\pi)$  to the map  $j_\pi u^N$ . In view of Theorem 4.4 the commutative diagram

$$\begin{array}{ccc} H_n(B\pi; \mathbb{Q}) & \xrightarrow[\quad = \quad]{j_\pi} & H_n(S_0(B\pi; \mathbb{Q})) \\ u_*^N \uparrow & & 0 \uparrow \\ H_n(N; \mathbb{Q}) & \xrightarrow{\phi} & H_n(S_0(B\pi^{(n-1)}); \mathbb{Q}) \end{array}$$

implies that  $u_*^N([N]) = 0$  in  $H_n(B\pi; \mathbb{Q})$ .

Therefore,  $u_*^N([N]) = u_*^{N'}([N']) = u_*^K([K]) = k(u_*^M([M])) = 0$  and, hence,  $M$  is rationally inessential.  $\square$

The groups  $\pi$  that admit a finite complex  $B\pi$  are called *geometrically finite*. We call a group  $\pi$  a *duality group* if there is an integer  $n$  such that  $H^i(\pi, \mathbb{Z}\pi) = 0$  for all  $i \neq n$  and  $H^n(\pi, \mathbb{Z}\pi)$  is torsion free as an abelian group [Br].

Let  $P$  be some property of groups. We say that a group  $\pi$  has property  $P$  *virtually* if it contains a finite index subgroup  $\pi'$  that satisfies the property  $P$ .

**5.4. Corollary.** *Let  $M$  be a totally non-spin closed orientable  $n$ -manifold,  $n \geq 5$ , with positive scalar curvature whose fundamental group  $\pi$  is a virtual geometrically finite duality RS-group. Then the Weak Gromov Conjecture holds for  $M$ , i.e.,  $\dim_{mc} \widetilde{M} \leq n - 1$ .*

*Proof.* Let  $M$  be such a manifold with the fundamental group  $\pi$ . Let  $\pi'$  be a geometrically finite duality group satisfying the RS conditions with  $(\pi : \pi') < \infty$ . Let  $p : M' \rightarrow M$  be a finite-to-one covering map that corresponds to  $\pi'$ . Then  $M'$  satisfies the conditions of Theorem 5.3 and hence is rationally inessential. The duality group property implies that the group  $H_n^{lf}(E\pi')$  is torsion free. Therefore,  $ec_*^\pi u_*([M']) = 0$  in  $H_n^{lf}(E\pi)$  where  $u = u^{M'}$  and  $ec_*^\pi$  is the equivariant coarsening homomorphism. Since for a lift  $\tilde{u}$  of  $u$  we have  $ec_*^\pi u_* = \tilde{u}_* ec_*^{M'}$  (see [Dr1] for this and the definition of  $ec$ ), we obtain  $\tilde{u}_*([\widetilde{M}']) = 0$ . Then by the Theorem 3.8,  $\dim_{mc} \widetilde{M}' \leq n - 1$ . Finally, we note that  $\widetilde{M} = \widetilde{M}'$ .  $\square$

**5.5. Remark.** There are examples of duality groups which are not geometrically finite [D3]. The virtual geometrically finite duality groups form a large class that includes virtually free groups, virtually nilpotent groups, arithmetic groups, mapping class groups,  $Out(F_n)$ , knot groups, and their products.

## 6. STRONG GROMOV'S CONJECTURE

We recall that the group of oriented relative bordisms  $\Omega_n(X, Y)$  of the pair  $(X, Y)$  consists of the equivalence classes of pairs  $(M, f)$  where  $M$  is an oriented  $n$ -manifold with boundary and  $f : (M, \partial M) \rightarrow (X, Y)$  is continuous map. Two pairs  $(M, f)$  and  $(N, g)$  are equivalent if there is a pair  $(W, F)$ ,  $F : W \rightarrow X$  called a bordism where  $W$  is an orientable  $(n+1)$ -manifold with boundary such that  $\partial W = M \cup W' \cup N$ ,  $W' \cap M = \partial M$ ,  $W' \cap N = \partial N$ ,  $F|_M = f$ ,  $F|_N = g$ , and  $F(W') \subset Y$ .

**6.1. Proposition.** *For any CW complex  $K$  there is an isomorphism*

$$\Omega_n(K, K^{(n-2)}) \cong H_n(K, K^{(n-2)}).$$

*Proof.* Since  $\Omega_1(*) = 0$  and  $K/K^{(n-2)}$  is  $(n-2)$ -connected, we obtain that in the Atiyah-Hirzebruch spectral sequence on the diagonal  $p+q = n$  there is only one nonzero term which survives to  $\infty$ :

$$E_{n,0}^2 \cong E_{n,0}^\infty \cong H_n(K, K^{(n-2)}; \Omega_0(*)) \cong H_n(K, K^{(n-2)}).$$

Therefore,

$$\Omega_n(K, K^{(n-2)}) \cong H_n(K, K^{(n-2)}). \quad (*)$$

□

**6.2. Proposition.** *Suppose that an  $n$ -manifold  $N$  is obtained from a manifold  $M$  by a chain of  $k$ -surgeries with  $k \geq 2$ . Assume that a classifying map  $u^N : N \rightarrow B\pi$  admits a deformation to  $B\pi^{(n-2)}$ . Then so does  $u^M : M \rightarrow B\pi$ .*

*Proof.* Let  $W$  be the bordism that corresponds the surgery. Then  $M$  is obtained from  $N$  by attaching handles of dimension  $n-k$ . Then the map  $u_n$  can be extended to a map  $g : W \rightarrow B\pi^{(n-2)}$ . Since the inclusion  $M \rightarrow W$  is a 2-equivalence, we obtain that the restriction  $g|_M$  induces isomorphism of the fundamental groups, and hence is a classifying map for  $\widetilde{M}$ . □

Let  $\nu_X : X \rightarrow BSO$  denote a classifying map for stable normal bundle of a compact manifold  $X$ .

**6.3. Lemma.** *Let  $M$  be a totally non-spin closed orientable inessential  $n$ -manifold,  $n \geq 5$ . Then a classifying map  $u^M : M \rightarrow B\pi$  can be deformed to  $B\pi^{(n-2)}$ , in particular,  $\dim_{mc} \widetilde{M} \leq n-2$ .*

*Proof.* We assume that a CW structure on  $M$  has one  $n$ -dimensional cell. In view of Lemma 3.5 there is a classifying map  $f : M \rightarrow B\pi$  with  $f(M \setminus D) \subset B\pi^{(n-2)}$  where  $D$  is an  $n$ -ball. Note that the restriction of  $f$  to  $(D, \partial D)$  defines a zero element in  $H_n(B\pi, B\pi^{(n-2)})$ . Therefore by Proposition 6.1 there is a relative bordism  $(W, q)$  of  $(D, \partial D)$  to  $(N, S)$  with  $q(N \cup \partial W \setminus D) \subset B\pi^{(n-2)}$ . We may assume that the bordism  $W' \subset \partial W$  of the boundaries  $\partial D \cong S^{n-1}$  and  $S$  is stationary,  $W' \cong \partial D \times [0, 1]$  and  $q(x, t) = q(x)$  for all  $x \in \partial D$  and all  $t \in [0, 1]$ . By performing 1-surgery on  $W$  we may assume that  $W$  is simply connected. Let  $\bar{W}$  be an extension of  $W$  to a bordism of  $M$  by the product bordism. Let  $i : M \rightarrow \bar{W}$  denote the inclusion map. Thus,  $i$  induces isomorphism of the fundamental groups.

Note that every 2-sphere  $S$  that generate an element of the kernel  $\ker(\nu_{\bar{W}})_*$  of  $(\nu_{\bar{W}})_* : \pi_2(\bar{W}) \rightarrow \pi_2(BSO)$ , has trivial stable normal bundle. It is easy to see that  $\pi_2(\bar{W})$  as a  $\pi$ -module is finitely generated. Hence we can apply surgery in dimension 2 on  $\bar{W}$  to obtain a map  $\nu_{\bar{W}} :$

$\bar{W} \rightarrow BSO$  that induces an isomorphism  $(\nu_{\bar{W}})_* : \pi_2(\bar{W}) \rightarrow \pi_2(BSO)$ . The assumption that  $\widetilde{M}$  is non-spin implies that  $(\nu_M)_* : \pi_2(M) \rightarrow \pi_2(BSO) = \mathbb{Z}_2$  is surjective. Since  $\nu_M = \nu_{\bar{W}} \circ i$  is an epimorphism and  $(\nu_{\bar{W}})_*$  is an isomorphism, it follows that  $i_* : \pi_2(M) \rightarrow \pi_2(\bar{W})$  is an epimorphism. Therefore,  $\bar{W}$  is obtained from  $D \times I$  by attaching disks of dimension  $\geq 2$  and thickening. Note that the bordism  $(\bar{W}, \bar{q})$  with the map  $\bar{q} : \bar{W} \rightarrow B\pi$  between  $(M, f)$  and  $(M', g)$  is obtained by a  $k$ -surgery for  $k \geq 2$  with  $g(M') \subset B\pi^{(n-2)}$ . Proposition 6.2 completes the proof.  $\square$

**6.4. Theorem.** *Gromov's Conjecture holds true for  $n$ -manifolds  $M$ ,  $n \geq 5$ , whose fundamental group is virtually the product of free groups. In particular it holds for virtually abelian groups.*

*Proof.* It suffices to consider the case when  $\pi_1(M)$  is the product of free groups.

Let  $M$  be totally non-spin. It was proven in [BD] that product of free groups satisfies the RS conditions.

$M$  is inessential. Then Lemma 6.3 implies that  $\dim_{mc} \widetilde{M} \leq n - 2$ .

We note that the product of free groups is a duality group. Also the coarse Baum-Connes conjecture holds true for the product of free groups [Yu]. Then the main result of [Dr1] implies the inequality  $\dim_{mc} \widetilde{M} \leq n - 2$  for almost spin  $M$ .  $\square$

**6.1. Further Refinements.** The second author modified Gromov's definition of the macroscopic dimension by imposing an additional restriction on the uniformly cobounded map  $f : X \rightarrow K$  to a simplicial complex to be Lipschitz [Dr2]. Here the metric on  $K \subset \ell_2(K^{(0)})$  is taken from the Hilbert space spanned by the vertices of  $K$  for any realization of  $K$  in the standard simplex. The new macroscopic dimension was denoted by  $\dim_{MC}$ . Clearly,  $\dim_{mc} \leq \dim_{MC}$ . It was shown that the rational inessentiality for an  $n$ -manifold  $M$  does not follow from the inequality  $\dim_{MC} \widetilde{M} < n$  [Dr2]. Also, it was shown that  $\dim_{mc} \neq \dim_{MC}$  [Dr3]. Thus, one can introduce an intermediate version of Gromov's conjecture asserting that  $\dim_{MC} \widetilde{M} \leq n - 2$  for the universal covering of a PSC  $n$ -manifold. In view of the main result of this paper and that of [Dr1] it is reasonable to expect that  $\dim_{MC} \widetilde{M} \leq n - 2$  for totally non-spin manifolds with (virtual) duality fundamental group satisfying the coarse Baum-Connes conjecture.

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